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How useful is commonality? Inventory and production decisions to maximise survival probability in start-ups

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Abstract

This paper deals with component commonality in start-up manufacturing firms. We present a two-product Markov decision model that examines the implications of the inventory and production strategies for the survival probability of the firm. The advantage of using component commonality is studied for varying costs, demand correlations, and order replenishment lead times. Optimal policies are derived, and minimum stock levels for survival are obtained. Moreover, we state the conditions under which simplified production decisions can be made. It is shown that commonality is not only useful as a way of dealing with demand uncertainty, but that its increased use is preferred under strong substitutability of products, and shorter replenishment lead times.

Keywords: Inventory; Manufacturing; Commonality; Stochastic Modeling; Dynamic programming.

1 Introduction

Start-up manufacturing companies play a vital role in the economy of any industrialised country and yet there is a high failure rate for such companies. It is important to identify strategies that ensure success for these newly formed, capital constrained companies. These are not necessarily the same strategies that are best for well established companies with access to large amounts of capital. The main difference is that start-up companies are primarily interested in the probability of survival as opposed to maximising the average profit, which is the criterion found in almost all previous management science models and is the one best suited to well established companies. Archibald, Thomas *et al* (2002) were the first to focus on the survival probability objective for small firms. Archibald (2002) and Possani (2003) both consider a one product model; they show how the inventory purchasing policies need to be more cautious for start-up companies than for well established firms that are not as restricted in capital. They also showed that as capital increases a start-up company behaves more like an established one. In this paper we expand on their ideas through an analysis of a two product model with commonality.

This paper looks at a simplified inventory and production problem which typifies the problems faced by many start-up manufacturing companies (specially those where the main profit is through adding value to bought in components). Given that cash flow is a major concern for these small firms, they are anxious not to have a large inventory of components, which suggests flexibility of components so that the same component could be use in more than one product. We investigate the impact of this commonality of components on the probability of survival. Our study is based on a two-product Markov decision model for inventory and production control, where one component is flexible and may be used to produce either of the two products. The objective is to find optimal inventory and production strategies for small firms, and investigate the impact the common component has on the probability of survival. These optimal policies are obtained and their behaviour is studied with changes in the ordering and sale costs, demand correlations, and order replenishment lead time. Although the model we analyse is a simple one, its aim is to provide a framework and give insight about the relationship between inventory, production, and financial decision in such a manufacturing environment.

Previous paper on commonality like Kim (2000), Desai (2001) and Ma (2002) concentrate mainly on the choice of depth of commonality to be used, as greater commonality result in less effort in product design and differentiation, as explained by Lee (1996). This is not the case for a start-up company where capital is a strong constraint, and where product diversity is not as extensive. However, commonality is one way of dealing with uncertainty in product demand. This is referred to in the literature as risk pooling. Baker (1986), Gerchark (1988) and recently Agrawal (2001) explore the effect on inventory levels due to risk pooling when common components are present and there are service levels requirements. In our model lost sales, which are a measure of service quality, are penalised by loss of revenue since any unsatisfied order is lost. Hillier (2001) also considers service level requirements in a multiple-product model with one common component that is more expensive than others. Eynan (1996) studied a commonality model with correlated demands and found that the impact of commonality is stronger with negative correlation. The results we obtain from our model support his findings. However, unlike any of the previous models we focus on the survival probability of the firm, considering finance, purchasing and production decisions at the same time.

In Section 2, we present the Markov decision model and derive some basic properties about the probability of survival in the long run. We show that some production decision can be simplified, and state the minimum inventory levels for survival. The advantage of commonality on the survival probability is explored in Section 3, where we give conditions under which the common component need not necessarily be employed. In Section 4 we analyse the impact varying costs and capital have on the production decisions in the zero lead time case. We give conditions under which full use of the common component is optimal, and analyse the behaviour of the production decisions with increases in inventory levels and capital. The case with 1 lead time is explored in Section 5, where we show that shorter replenishment lead times may improve the probability of survival, and are more likely to lead to commonality; some examples for varying costs and capital are displayed. Our conclusions are summarised in Section 6.

2 Survival Probability Model

Consider a firm which manufactures two type of products, from components that have to be purchased from other manufacturers. The company has two types of components from which to manufacture the products. Component 1 is a unique component and is used exclusively to produce product 1, whereas component 2 is a common (flexible) component and may be employed to manufacture both products. The demand for the products is random with independent identical distributions each time period. We assume products cannot be manufactured if components are not available. We consider the lead time for ordering components as fixed, and analyse both a zero and one lead time for the arrival of orders. If the demand in a period exceeds the number of components available in stock for production then the excess demand is lost. Ordering too many components ties up the firm's capital in stock which is not required, whereas ordering too few leads to unsatisfied demand and hence a loss of profit. The decision the firm faces is how many components to order each period, and how many components of each type to employ in manufacturing the products. In the one lead time case these assumptions really mean that the manufacturing time and any variation in lead time is small compared with the lead time itself. Figure 1 shows a time line for the events and decisions in one period for a one lead time delay in the arrival of orders. In the zero lead time case the orders are received in a short period of time, however the demand is not known until after this orders arrive. At the end of each period, the firm's capital is checked and if it is negative, the firm ceases to exist. Figure 2 shows a time line for the events and decisions in the zero lead time case.

Figure 1: A time line for the events in one period for 1 lead time

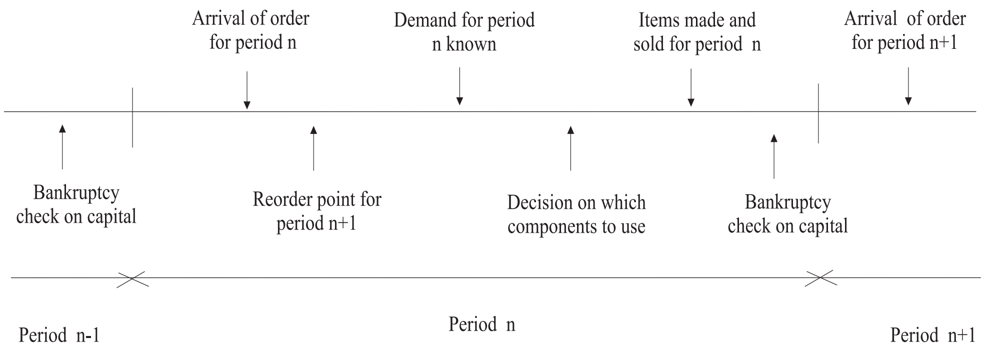
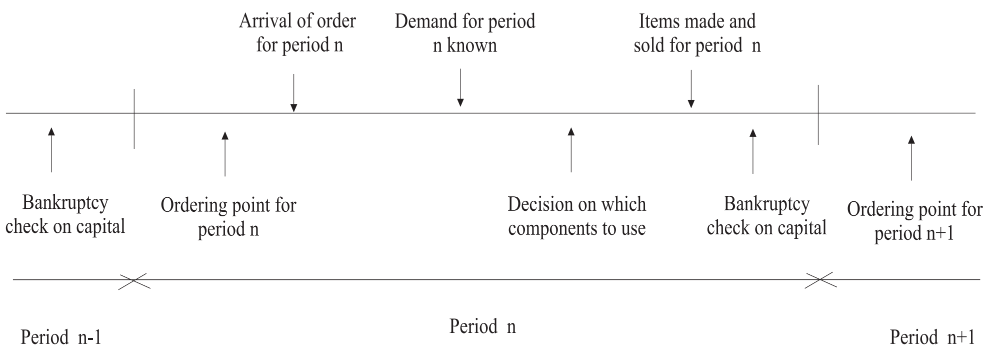


Figure 2: A time line for the events in one period for zero lead time



Let C_j be the purchasing price of component j, and S_j be the sale price of product j $(j = 1, 2)$. We assume $S_2 \geq S_1$, and as component 2 is more flexible we expect it to have a higher cost than component 1 (i.e $C_2 \geq C_1$). There is a fixed overhead cost of H per period (e.g. cost of staff and premises which are incurred irrespective of the activity of the firm). The probability of having a demand of d_1 of product 1, and d_2 of product 2 is $p(d_1, d_2)$. Let $M_j = \max\{d_j \mid p(d_1, d_2) > 0\}$ be the maximum possible demand of product j ($j = 1, 2$) that can be satisfied in a period (this can be interpreted as the maximum production capacity in a period), and let $\overline{d_j}$ be the average demand of product j (i.e. $\overline{d_j} = \sum_{d_1, d_2} d_j p(d_1, d_2)$) for $j = 1, 2$. We will assume throughout the paper that $(S_1 - C_1)\overline{d_1} + (S_2 - C_2)\overline{d_2} > H$, so that in the long run the firm can be profitable.

As the company is a newly set-up firm it has limited capital, and will not survive if it uses all of it. Hence, at any point the state of the firm is described by three variables, x the capital, i_1 the number of components of type 1 in stock, and i_2 the number of components of type 2 in stock. Let j_{hl} be the number of products h made using component l in that period. Define $q(n, i_1, i_2, x)$ as the maximum probability that a firm will survive n more periods given that it has i_1 components of type 1 ($i_1 \geq 0$), i_2 components of type 2 ($i_2 \geq 0$) in stock and x units of capital. We assume that storage constraints put some upper limit on i_1 and i_2, and so we have a finite action space since we cannot order more than this amount. $q(n, i_1, i_2, x)$ is the optimal function for a finite horizon dynamic programming problem with a countable state space (x assumed to have discrete levels), and a finite action space - the amount k_1 to order of the first component , k_2 the amount to order of component 2, and the production decisions j_{11}, j_{22}, j_{12}. Thus, it has an optimal non-stationary policy, see Puterman (1994). Moreover, the survival probability $q(n, i_1, i_2, x)$ satisfies the following dynamic optimality equation

$$
\begin{aligned}
q(n, i_1, i_2, x) = \quad & \max_{k_1, k_2} \Big\{ \sum_{d_1, d_2} p(d_1, d_2) \max_{j_{11},\, j_{22},\, j_{12}} q(n-1,\ i_1 + k_1 - j_{11},\ i_2 + k_2 - j_{22} - j_{12}, \\
& \qquad x + S_2 j_{22} + S_1(j_{11} + j_{12}) - C_1 k_1 - C_2 k_2 - H) \Big\},
\end{aligned} \tag{1}
$$

where $0 \leq j_{11} \leq J_{11}$, $0 \leq j_{22} \leq J_{22}$, and $0 \leq j_{12} \leq J_{12}$, with boundary conditions $q(n, i_1, i_2, x) = 0$ $\forall x < 0$, and $q(0, i_1, i_2, x) = 1 \ \forall\ i_1, i_2, x \geq 0$. For the one lead time case $J_{11} = \min\{i_1,\ d_1\}$, $J_{22} = \min\{i_2,\ d_2\}$, and $J_{12} = \min\{d_1 - j_{11},\ i_2 - j_{22}\}$; whereas for the zero lead time case $J_{11} = \min\{i_1 + k_1,\ d_1\}$, $J_{22} = \min\{i_2 + k_2,\ d_2\}$, and $J_{12} = \min\{d_1 - j_{11},\ i_2 + k_2 - j_{22}\}$. Note that if the second component cannot be used to substitute the first component (no commonality) then $J_{12} = 0$, while J_{11}, and J_{22} are as above. Define $k_1(n, i_1, i_2, x)$, and $k_2(n, i_1, i_2, x)$ as the values for k_1 and k_2 respectively where the maximum is reached in the right hand side of the equation for $q(n, i_1, i_2, x)$ above. In other words, $k_1(n, i_1, i_2, x)$, $k_2(n, i_1, i_2, x)$, is the optimal ordering policy for given inventory levels i_1, i_2 and capital x. If we assume the limit of $q(n, i_1, i_2, x)$ as $n \to \infty$ exists, the limit $q(i_1, i_2, x) = \lim_{n\to\infty} q(n, i_1, i_2, x)$ is the probability that the firm will survive forever given that it has i_1 components of the first type and i_2 of the second type in stock, and x units of capital.

There are some basic properties which one expects of the survival probability $q(n, i_1, i_2, x)$ as n, i_1, i_1, and x vary and these are confirmed in the following Lemma.

Lemma 1

i) $q(n, i_1, i_2, x)$ is non-increasing in n.

ii) $q(n, i_1, i_2, x)$ is non-decreasing in x.

iii) $q(n, i_1, i_2, x)$ is non-decreasing both in i_1 and in i_2.

Proof:

The proof is by induction on n. Since $q(n, i_1, i_2, x) = 0$ when $x < 0$ for all $i_1, i_2 \geq 0$ and for all n; $q(0, i_1, i_2, x) = 1$ when $x \geq 0$ for all $i_1, i_2 \geq 0$; and $q(1, i_1, i_2, x) \leq 1$ when $x \geq 0$, all three hypotheses hold in the case $n = 0$.

Let $i_1' = i_1 + k_1 - j_{11}$, $i_2' = i_1 + k_2 - j_{12} - j_{22}$, $x' = x + S_2 j_{22} + S_1(j_{11} + j_{12}) - C_1 k_1, C_2 k_2 - H$, assume all three hypotheses hold for n, and use $\max_i\{a_i\} - \max_i\{b_i\} \leq \max_i\{a_i - b_i\}$ to show

i) $q(n+1, i_1, i_2, x) - q(n, i_1, i_2, x) \leq$

$$\max_{k_1,k_2}\Big\{ \sum_{d_1,d_2} p(d_1,d_2)[\max_{j_{11},\, j_{12},\, j_{22}} q(n, i_1', i_2', x') - \max_{j_{11},\, j_{12},\, j_{22}} q(n-1, i_1', i_2', x')] \Big\} \leq$$

$$\max_{k_1,k_2}\Big\{ \sum_{d_1,d_2} p(d_1,d_2) \max_{j_{11},\, j_{12},\, j_{22}} \big(q(n, i_1', i_2', x') - q(n-1, i_1', i_2', x') \big) \Big\} \leq 0$$

Hence, hypothesis (i) holds for $n+1$.

ii) $q(n+1, i_1, i_2, x) - q(n+1, i_1, i_2, x+a) \leq$

$$\max_{k_1,k_2}\Big\{ \sum_{d_1,d_2} p(d_1,d_2) \max_{j_{11},\, j_{12},\, j_{22}} \big(q(n, i_1', i_2', x') - q(n, i_1', i_2', x'+a) \big) \Big\} \leq 0 \text{ (where } a > 0).$$

Hence, hypothesis (ii) holds for $n+1$.

iii) $q(n+1, i_1, i_2, x) - q(n+1, i_1+1, i_2, x) \leq$

$$\max_{k_1,k_2}\left\{ \sum_{d_1,d_2} p(d_1,d_2)[\max_{j_{11},\, j_{12},\, j_{22}} q(n, i_1', i_2', x') - \max_{j_{11},\, j_{12},\, j_{22}} q(n, i_1'+1, i_2', x')\,] \right\}$$

note that in this case, the ranges for j_{11}, j_{12}, j_{22} are not the same for the two innermost maximisation functions. However as $\min\{i_1(+k_1), d_1\} \leq \min\{i_1(+k_1)+1, d_1\}$, the first set is contained in the second set. Increase the range of the first set to be $0 \leq j_{11} \leq \min\{i_1(+k_1)+1, d_1\}$, this will not reduce the value of the first innermost max function and so

$q(n+1, i_1, i_2, x) - q(n+1, i_1+1, i_2, x) \leq$

$$\max_{k_1,k_2}\Big\{ \sum_{d_1,d_2} p(d_1,d_2) \max_{j_{11},\, j_{12},\, j_{22}} \big(q(n, i_1', i_2', x') - q(n-1, i_1'+1, i_2', x') \big) \Big\} \leq 0$$

A similar proof holds for i_2. Hence, hypothesis (iii) holds for $n+1$.

□

Lemma 2 (We prefer capital to inventory)

i) For all i_1 $q(n, i_1+b, i_2, x) \leq q(n, i_1, i_2, x + S_1 b)$ $\forall\, b, x > 0$, and

ii) For all i_2 $q(n, i_1, i_2+b, x) \leq q(n, i_1, i_2, x + S_2 b)$ $\forall\, b, x > 0$.

Proof:

i) Suppose the optimal purchasing and production policy starting in state (n, i_1+b, i_2, x) is $\pi^* = (k_1^*(t, h_t),\, k_2^*(t, h_t),\, j_{11}^*(t, h_t), j_{12}^*(t, h_t), j_{22}^*(t, h_t))$, $t = 1, \ldots, n$, where h_t is the history up to time t, h_t describes the choice of k_1, k_2, the realized demand d_1, d_2 and the production decisions j_{11}, j_{22}, j_{12} for all the first $t-1$ periods. Let policy π' be exactly the same as π^* except that the first component is not used to produce product 1 (i.e. $j'_{11} = 0$) until b type 1 products have been produced using components of type 1 under π^*. Let $n'(h_{n'})$ be the period such that $\sum_{t=1}^{n'(h_{n'})} j^*_{11}(t, h_t) > b$, where $j^*_{11}(t, h_t)$ is the optimal production policy at period t with history h_t.

Note that both π^* and π' buy the same amount of components, but that π' uses b less component of type 1 up to period n' to manufacture product 1. So that $j'_{11}(n'(h_{n'}), h_{n'}) = j^*_{11}(n'(h_{n'}), h_n) - b$, and $j'_{11}(n, h_n) = j^*_{11}(n, h_n)$ for $n > n'(h_{n'})$. Recall that S_1 is the selling price of product 1; if $n'(h_{n'}) > n$ we have that $q^{\pi^*}(n, i_1 + h, i_2, x) \leq q^{\pi'}(n, i_1, i_2, x + S_1 h)$, where q^π is the probability of survival under policy π. The inequality holds since π' will always have more cash available than π^*. On the other hand if $n'(h_{n'}) \leq n$, that is after $n'(h_{n'})$ periods have passed, both policies have the same number of components and the same amount of capital. Not only that, but both policies are the same after that point. Hence, their subsequent chances of failure is the same and so

$$q^{\pi^*}(n, i_1 + b, i_2, x) \leq q^{\pi'}(n, i_1, i_2, x + S_1 b).$$

Note that π' is a valid policy so that $q^{\pi'}(n, i_1, i_2, x + S_1 b) \leq q(n, i_1, i_2, x + S_1 b)$. On the other hand π^* is the optimal policy for $(n, i_1 + b, i_2, x)$ so $q(n, i_1 + b, i_2, x) = q^{\pi^*}(n, i_1 + b, i_2, x)$ thus

$$q(n, i_1 + b, i_2, x) \leq q(n, i_1, i_2, x + S_1 b),$$

so property i) holds for n.

ii) Using similar arguments, suppose $\pi^* = (k^*_1(t, h_t), k^*_2(t, h_t), j^*_{11}(t, h_t), j^*_{12}(t, h_t), j^*_{22}(t, h_t))$ is the optimal purchasing and production policy ($t = 1, \ldots, n$) from state $(n, i_1, i_2 + b, x)$ where h_t is the history of the first $t - 1$ of the n periods. Consider a policy π' exactly the same except that the second component is not used (i.e. $j_{12} = j_{22} = 0$) until a period $n'(h_{n'})$ such that $\sum_{t=0}^{n'(h_{n'})} j^*_{12}(t, h_t) + j^*_{22}(t, h_t) > b$, where $j^*_{12}(t, h_t)$ and $j^*_{22}(t, h_t)$ are the optimal production policy for the second component at period t with history h_t. Under such conditions we have that

$$q(n, i_1, i_2 + b, x) = q^{\pi^*}(n, i_1, i_2 + b, x) \leq q^{\pi'}(n, i_1, i_2, x + S_2 b) \leq q(n, i_1, i_2, x + S_2 b),$$

where q^π is the probability of survival under policy π. Hence property ii) holds for n.

□

These results are valid for the infinite horizon $q(i, x)$, as described in the following Lemma.

Lemma 3

i) $q(i_1, i_2, x) = \lim_{n \to \infty} q(n, i_1, i_2, x)$ *exists.*

ii) For all i, $q(i_1 + b, i_2, x) \leq q(i_1, i_2, x + S_1 b)$, *and* $q(i_1, i_2 + b, x) \leq q(i_1, i_2, x + S_2 b)$.

iii) $q(i_1, i_2, x)$ *is non-decreasing in* x.

iv) $q(i_1, i_2, x)$ *is non-decreasing in* i_1, *and* i_2.

Proof:

$q(n, i_1, i_2, x)$ is bounded above by 1 and below by 0, and from Lemma 1 i) is monotonic non-increasing in n. As bounded monotonic sequences converge, property i) follows. Property ii), follows immediately by taking the limit in the results of Lemma 2, while, properties iii), and iv) follow by taking the limit in results of Lemma 1 ii) and iii).

□

Our subsequent results hold for both the finite and the infinite horizon cases. We shall normally prove them in the infinite horizon context but the proofs go through in the finite horizon case also.

Let us now focus on the production decisions j_{lh}. We say that a component is fully utilized if $j_{ll} = J_{ll}$ $(l = 1, 2)$. Similarly we say there is full substitution if $0 < j_{12} = J_{12}$. We can prove that it is always optimal to fully utilize components j_{11} and j_{22} as follows.

Theorem 1 (Full utilization) *The production decision $j_{ll} = J_{ll}$ is optimal.*

Proof:

Consider first the one lead time case. Suppose we use less than $J_{ll} = \min\{i_l, d_l\}$ components of type l to satisfy the demand for product l. For $l = 1$ suppose we use one less item than J_{11}, this leads to the state $(\delta+1, i_2, x)$ rather than $(\delta, i_2, x+S_1)$, but from Lemma 3 ii) $q(\delta, i_2, x+S_1) \geq q(\delta+1, i_2, x)$. Hence, it is better to use that component to make one more item of product 1. For $l = 2$ suppose we make one less than J_{22} components of type 2. The extra component of type 2 can be left in stock which leads to state $(i_1, \delta + 1, x)$ or used to make one more item of product 1, which leads to state $(i_2, \delta, x + S_1)$. From Lemma 3 ii) and iii) it follows that $q(i_1, \delta, x + S_2) \geq q(i_1, \delta, x + S_1)$, and $q(i_1, \delta, x + S_2) \geq q(i_1, \delta + 1, x)$, and so it is best to make J_{22} products of type 2. Lemma 3 also holds in the zero lead time case, and an identical argument proves the result in this case.

□

Theorem 2 (Minimum ordering quantity for survival)
There is a minimum ordering boundary for the ordering policy (k_1, k_2) when $i_1 = i_2 = 0$, , namely $(S_1 - C_1)k_1 + (S_2 - C_2)k_2 \geq H$, such that if $k_1(0,0,x)$, and $k_2(0,0,x)$ are not within the boundary then $q(0,0,\hat{x}) = 0$ for all $\hat{x} \leq x$.

Proof:
Let $(k_1, k_2) \in L$ if $(S_1 - C_1)k_1 + (S_2 - C_2)k_2 < H$, and $(k_1, k_2) \in U$ if $(S_1 - C_1)k_1 + (S_2 - C_2)k_2 \geq H$. Assume there are no optimal component purchasing policies (k_1, k_2) for $(0, 0, x)$ with $(k_1, k_2) \in U$. Let the optimal component purchasing policies for state $(0, 0, x)$ be $k_1(0, 0, x) = \alpha_1$, and $k_2(0, 0, x) = \alpha_2$, with $(\alpha_1, \alpha_2) \in L$. Define $\epsilon = H - (S_1 - C_1)\alpha_1 + (S_2 - C_2)\alpha_2$; and note that $\epsilon > 0$. Thus we have that,

$$q(0,0,x) = q(\alpha_1, \alpha_2, x - C_1\alpha_1 - C_2\alpha_2 - H) \leq q(0,0,x + (S_1 - C_1)\alpha_1 + (S_2 - C_2)\alpha_2 - H) = q(0,0,x-\epsilon),$$

where the first inequality holds from Lemma 3 ii). We now show there is no optimal component purchasing policy for state $(0, 0, x - \epsilon)$ with $(k_1, k_2) \in U$. Assume the opposite and let the optimal component purchasing policies for state $(0, 0, x - \epsilon)$ be $k_1(0, 0, x - \epsilon) = \delta_1$, and $k_2(0, 0, x - \epsilon) = \delta_2$ with $(\delta_1, \delta_2) \in U$, then

$$q(0,0,x) > q(\delta_1, \delta_2, x - C_1\delta_1 - C_2\delta_2 - H) \geq q(\delta_1, \delta_2, x - \epsilon - C_1\delta_1 - C_2\delta_2 - H) = q(0,0,x-\epsilon),$$

where the first inequality follows from the assumption that there is no optimal purchasing policy in $(0,0,x)$ with $(k_1,k_2) \in U$, and the second inequality follows from Lemma 3 iii). This contradicts that $q(0,0,x) \leq q(0,0,x-\epsilon)$, and so $(\delta_1,\delta_2) \in L$.

Repeating the argument shows that $q(0,0,x-\epsilon) \leq q(0,0,x-2\epsilon) \leq \ldots \leq q(0,0,x-n\epsilon) = q(0,0,y)$. If we make $n \geq x/\epsilon$ then $y < 0$ and so $q(0,0,x) = 0$. Hence, Lemma 3 iii) implies that $q(0,0,\hat{x}) = 0$ for all $\hat{x} \leq x$.

□

Hence, there is lower bound on the amount of capital a small firm must start with to be able to order enough items to survive in the long run.

3 Commonality and correlation

Having a common component (in our model this is component 2) which can be utilized to make both products cannot have a negative effect on the chance of the firm surviving. Theorem 3 formalises this obvious result, and Example 1 shows commonality would lead to significant improvements in survival probability. However, the common component need not always prove an advantage as is shown in Theorem 4. It seems that the advantage of using commonality relies in part on the correlation of the demand for products.

Theorem 3

Let q_{nc} be the probability of survival when commonality is not present (i.e. where $J_{12} = 0$), then if q is the probability of survival when commonality is present (i.e. $J_{12} \geq 0$) then

$$q(n,,i_1,i_2,x) \geq q_{nc}(n,,i_1,i_2,x).$$

Proof:

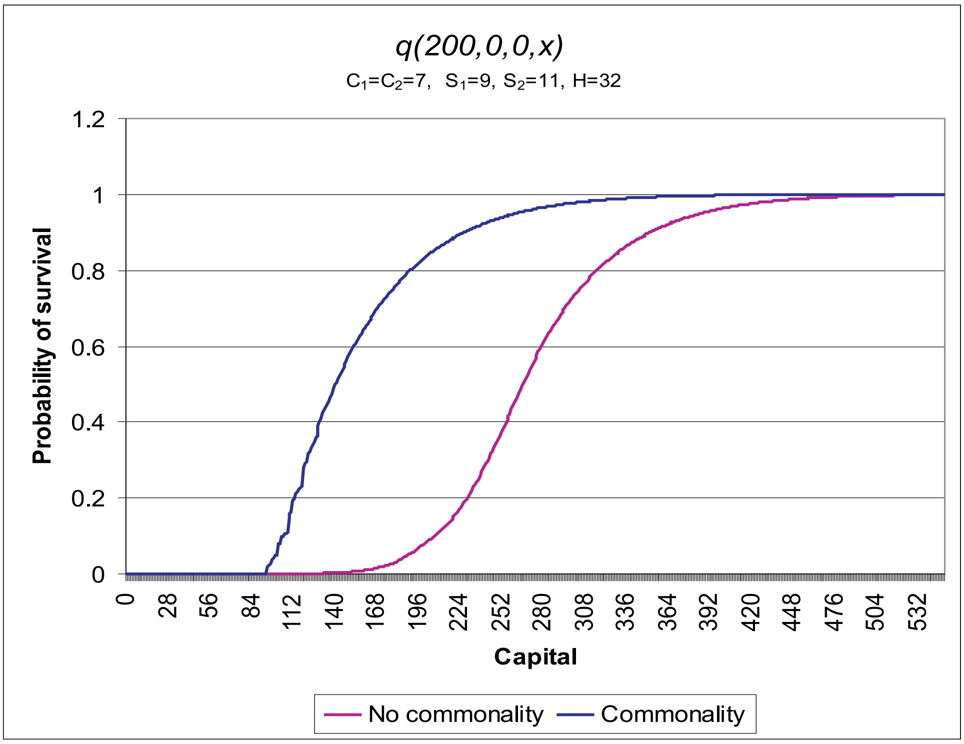
From the optimality equation one can see that a situation without commonality, that is $J_{12} = 0$, is contained within the decision when commonality is permitted (i.e for the zero lead time case $J_{12} = \min\{d_1 - j_{11}, i_2 + k_2 - j_{22}\}$, and one lead time $J_{12} = \min\{d_1 - j_{11}, i_2 - j_{22}\}$). Hence commonality is better than no commonality.

□

Example 1.- Advantage of a common component

Compare a situation where each component can only make one product (i.e. component 1 can only make product 1 and component 2 only product 2), with a situation where component 2 could be used to make both products: 1 and 2. Take $C_1 = C_2 = 7$, $S_1 = 9$, $S_2 = 11$, $H = 32$, and let the demand distribution be $p(0,11) = 0.5$, and $p(11,0) = 0.5$ (negatively correlated). Figure 3 shows the probability of survival for these situations. We graph results for $n = 200$, $i_1, i_2 = 0$, and $0 \leq x \leq 550$. We can see that there is considerable advantage in having a common component even if the cost of both components is the same. For example, note that for a capital level of $x = 190$ there is an 80% chance of survival when commonality is present whereas there is only a 5% chance of survival when there is no commonality.

Figure 3: Probability of Survival with and without commonality $q(200, 0, 0, x)$



This example also shows that commonality gives the same probability of survival with a smaller amount of capital than would be attainable without commonality. For example to have a 60% chance of survival we would only need $x = 153$ capital with commonality, where as we would need $x = 280$ without commonality.

Theorem 4 *If we have a completely correlated demand, identical component costs and product sale prices, starting with no inventory ($i_1 = i_2 = 0$), the optimal policy need never use commonality. In other words, under these conditions we can allow $J_{12} = 0$.*

Proof:

Assume that $C_1 = C_2 = C$, $S_1 = S_2 = S$, and that the overhead cost is H. The proof is as follows:

a) We first show that the model with two products is identical to a one product model with purchasing cost C, sale price S, and overhead cost H. We do this by producing a mapping of states from $(n, i_1, i_2, x) \to (n, i_1 + i_2, x)$, and policies $\pi_2 \to \pi_1$ so that

$$q_2^{\pi_2}(n, i_2, i_2, x) = q_1^{\pi_1}(n, i_1 + i_2, x),$$

where q_i^π is the probability of survival with i products and policy π $(i = 1, 2)$.

Let q_2 be the probability of survival with 2 products (i.e. the model presented in this paper), and q_1 the probability of survival for a one product model (as presented in Archibald (2002)). We can write the completely correlated demand as $p(d_1, d_2) = \begin{cases} 0 & \text{if } d_1 \neq d_2 \\ p'(2d_1) & \text{if } d_1 = d_2 \end{cases}$. Consider any policy for 2 products $\pi_2 : (k_1, k_2, j_{11}, j_{12}, j_{22})$ in state (n, i_1, i_2, x), and translate it into a 1 product policy $\pi_1 : (k, j)$ in state $(i_1 + i_2, x)$, where $k = k_1 + k_2$, and $j = j_{11} + j_{12} + j_{22}$. The effect on profit is the same, and so $q_2(i, i, x) = q_1(2i, x)$, and $q_2(i + 1, i, x) = q_1(2i + 1, x)$. Hence there is a mapping from the 2 product model to the 1 product model.

b) We now show that the 1 product model is equivalent to a 2 product problem where the inventory levels are always (i, i) or $(i + 1, i)$, when mapping states and policies into one another.

To translate from a 1 product policy (k, j) to a 2 product policy, consider first applying policy (k, j) in state $(2i, x)$. State $(2i, x)$ translates into the two product model in state (i, i, x), and translate policy (k, j) into the following policies: $(k/2, k/2, j/2, j/2)$ if k and j are even; $(k/2, k/2, \frac{j-1}{2}, \frac{j+1}{2})$ if k is even and j odd; $(\frac{k+1}{2}, \frac{k-1}{2}, j/2, j/2)$ if k odd and j are even; finally $(\frac{k+1}{2}, \frac{k-1}{2}, \frac{j+1}{2}, \frac{j-1}{2})$ if both k and j are odd. On the other hand, for the one product policy (k, j) in state $(2i + 1, x)$ this translates to two product policy for the state $(i + 1, i, x)$ as follows: $(k/2, k/2, j/2, j/2)$ if k and j are even; $(k/2, k/2, \frac{j+1}{2}, \frac{j-1}{2})$ if k is even and j odd; $(\frac{k-1}{2}, \frac{k+1}{2}, j/2, j/2)$ if k odd and j are even; finally $(\frac{k-1}{2}, \frac{k+1}{2}, \frac{j-1}{2}, \frac{j+1}{2})$ if both k and j are odd. Under such mapping the effect on the profit is the same, and the new states next period is always (i, i, x) or $(i + 1, i, x)$. Restricting ourselves only to allow states (i, i, x) or $(i + 1, i, x)$ does not change the probability of survival. Hence the 1 product model is equivalent to the 2 product one.

c) Finally we note that in state (i, i) or $(i + 1, i)$ there is no possibility of commonality because we never have extra demand for product 1 and spare inventory of product 2. Hence, the optimal policy must have no commonality and satisfies the optimality equation for the original problem, and must be optimal for it.

□

Recall that the Example 1 showed that a negatively correlated demand encourages commonality. We have just proven that a completely positively correlated demand with identical costs and sale prices does not need it at all. This suggest that commonality should be used when there is a strong substitution effect of the products which lead to negative correlations.

Note that the costs of both components are the same in Example 1 and Theorem 4. To address the question of the behaviour when there are differences in the purchasing costs of the two components we need to separate out the zero and the one lead time cases. We address this in detail in the next two sections.

4 Zero lead time with general costs structure

First let us analyse the case where there is still a profit to be made when the first product is made from component 2 (as well as component 1).

Theorem 5 (Full substitution optimal) *If* $S_1 \geq C_2$ *then* $j_{12} = J_{12}$.

Proof: (by contradiction)

Consider an optimal policy π^* , where $j^*_{12}(t)$ is the optimal production policy in period t, and suppose that at a given period t: $j^*_{12}(t) < J_{12}$. Let policy π' be the same as π^* except that the optimal production policy in period t under π' is $j'_{12}(t) = J_{12}$, and the ordering policy of the second component at period t under π' is $k'_2(t) = k^*_2(t) + J_{12} - j^*_{12}(t)$, where $k^*_2(t)$ is the optimal ordering policy for component 2 under policy π^* at period t. Then, as there is a zero lead time, at the end of period t we have the same number of items in inventory; not only that, but policies π' and π^* are the same thereafter. However, we have $(S_1 - C_2)(J_{12} - j^*_{12}(t))$ more capital under policy π' than π^*. Thus, $j^*_{12}(t)$ cannot be optimal. Hence, at any period t: $j_{12}(t) = J_{12}$.

$\square$

Corollary 1 (Monotonicity on substitution) *If* $S_1 \geq C_2$ *then* j_{12} *optimal is non-increasing in* i_1*, and non-decreasing in* i_2.

Proof:

It follows from Theorem 1, Theorem 5, and the definition of J_{12} in zero-lead time.

$\square$

Hence, in the one lead time case when there is a profit to be made by substituting the cheaper unique component 1 with component 2, we will always fulfill the demand for product 1 as much as possible. Not only that, but we will increase this substitution the more type of components 2 we have, and the more capital we have. Now let us focus in the cases where there is a loss of profit when we substitute component 1 with component 2.

Example 2.- Full substitution is not necessarily optimal when $S_1 < C_2$

Suppose that $p(d_1, d_2) = p'(d_1)p'(d_2)$ where $p'(d) \sim$ Poisson(5.5), $M_1 = M_2 = 20$, and $H = 38$. Let $C_1 = 1$, $S_1 = 3$, $C_2 = 10$, $S_2 = 15$. Note that under this conditions the firm does not makes a profit selling product 1 when component 2 is utilised to manufacture it (as $S_1 < C_2$), however the firm might avoid bankruptcy by selling the product to have cash to pay overheads (see Example 2) . Under this conditions the optimal purchasing policy for state $(2, 0, 0, 23)$ are: $k_1(2, 0, 0, 23) = 7$, $k_2(2, 0, 0, 23) = 5$; and the survival probability is $q(2, 0, 0, 23) = 0.666468$. Consider the case when $d_1 \geq 8$ and $d_2 = 4$. That is, $j_{11} = i_1 + k_1 = 7$, $j_{22} = d_2 = 4$, and $0 \leq j_{12} \leq 1$. The optimal decision is $j_{12} = 0$ and not $J_{12} = \min\{d_1 - j_{11},\ i_2 + k_2 - j_{22}\} = 1$, the firm should not substitute component 1 with component 2. This can be explained because under full substitution we will move to a state with $i_1 = i_2 = 0$ and $x = 12$, and with no substitution to a state with $i_1 = 0$, $i_2 = 1$ and $x = 9$, but $q(1,0,0,12) = 0.72775$, whereas $q(1,0,1,9) = .887407$. Thus it is better to keep the common component in inventory.

This example shows that the property in Theorem 5 does not extend for instances where $S_1 < C_2$.

Example 3.- Substitution will occur even if $S_1 < C_2$

One may think from the previous example that commonality is not useful when $S_1 < C_2$ as there is no profit to be made by manufacturing product 1 with component 2. We now show that is not the case. Consider the same instance as in Example 1 where $p(d_1, d_2) = p'(d_1)p'(d_2)$ and $p'(d) \sim$ Poisson(5.5), $M_1 = M_2 = 20$, $C_1 = 1$, $S_1 = 3$, $C_2 = 10$, $S_2 = 15$ and $H = 38$. Suppose $n = 2$, $i_1 = 0$, $i_2 = 0$, and $x = 1$, then $q(2,0,0,1) = 0.18632$, $k_1(2,0,0,1) = 7$, and $k_2(2,0,0,1) = 7$. Suppose $d_1 \geq 8$, and $d_2 \leq 6$, in particular look at a situation where $d_1 = 8$, and $d_2 = 6$, then $j_{11} = 7$ and $j_{22} = 6$. Substituting in full (i.e. $j_{12} = 1$) leads to a situation where we sell $(j_{11} + j_{12} =)$ 8 items of product 1, and $j_{22} = 6$ items of product 2, to obtain revenues of $114 = 8S_1 + 6S_2$, discounting the costs $H + C_1k_1 + C_2k_2 = 115$ and adding $x = 1$ leaves 0 unit of capital. As $q(1,0,0,0) = 0.376065$ we would manage to survive with a positive probability. However, if we did not substitute in full, that is if $j_{12} = 0$, then we would end up with a negative capital of $-3 = x - H - C_1k_1 - C_2k_2 + S_1(j_{11} + j_{12}) + S_2(j_{22})$, and thus the company would be bankrupt (recall that $q(1,0,1,-3) = 0$).

Example 3 shows that full substitution may be optimal for a specific instance even if $S_1 < C_2$. More important, it shows that we might need to sacrifice some profits in the short run to be able to survive in the long run. However, if we have enough capital the advantage of substituting at a loss of profit (as shown in Example 2) disappears as proven in the following Theorem.

Theorem 6 *If $S_1 < C_2$ there exists an x^* such that $j_{12} = 0$ in state (n, i_1, i_2, x) for $x > x^*$.*

Proof:

Suppose that at a given point t^* $j_{12} > 0$ (using component 2 to manufacture product 1) is an optimal decision for a state with inventory levels i_1, i_2 and capital x, and consider the optimal policy there after to be π^*, where the optimal ordering decision for the second component in period t is $k_2^*(t, h_t)$, given the history h_t is. Let period $n'(h_{n'})$ be such that $\sum_{t=1}^{n'(h_{n'})} k_2^*(t, h_t) > j_{12}$ (there is such an $n'(h_{n'})$ as the demand for product 2 is positive). Let us compare this policy with one, say π', where we do not substitute; that is, where $j'_{12} = 0$. First, let policy π' be exactly the same as π^* except that the ordering policy for the second component is $k'_2(t^*) = 0, \ldots, k'_2(t^* + n'(h_{n'}) - 1) = 0$ and $k'_2(t^* + n'(h_{n'})) = \sum_{t=1}^{n'(h_{n'})} k_2^*(t^* + t, h_{t^*}) - j_{12}$. Then, in period $n'(h_{n'})$ we have the same number of components in inventory under both policies. Not only that, but policies π' and π^* are the same from $n'(h_{n'})$ onwards. However, under policy π^* we have $x' + (C_2 - S_1)j_{12}$ capital, where x' is the capital under π', and as $S_1 < C_2$ we have less capital under π^*. So provided that we had at least an initial capital $x > x^*$ that will allow us to survive under π' for those n' periods (e.g. $x^* = Hn'(h_{n'}) + C_1 \sum_{t=1}^{n'(h_{n'})} k'_1(t^* + t, h_{t^*})$), from Lemma 1 ii) the survival probability is higher under

π' than under π^*, and thus $j_{12} > 0$ is not optimal. Thus, for $x > x^*$ (large enough) $j_{12} = 0$ is optimal.

□

5 One lead time with general cost structure

When turning to the one lead time case the first thing to note is that the increase in lead time decreases the probability of the firm to survive as stated in the following lemma.

Lemma 4 *If q_0 is the probability of survival in the zero lead time and q_1 is the probability of survival in the one lead time then*

$$q_0(n, i_1, i_2, x) \leq q_1(n, i_1, i_2, x).$$

Proof:

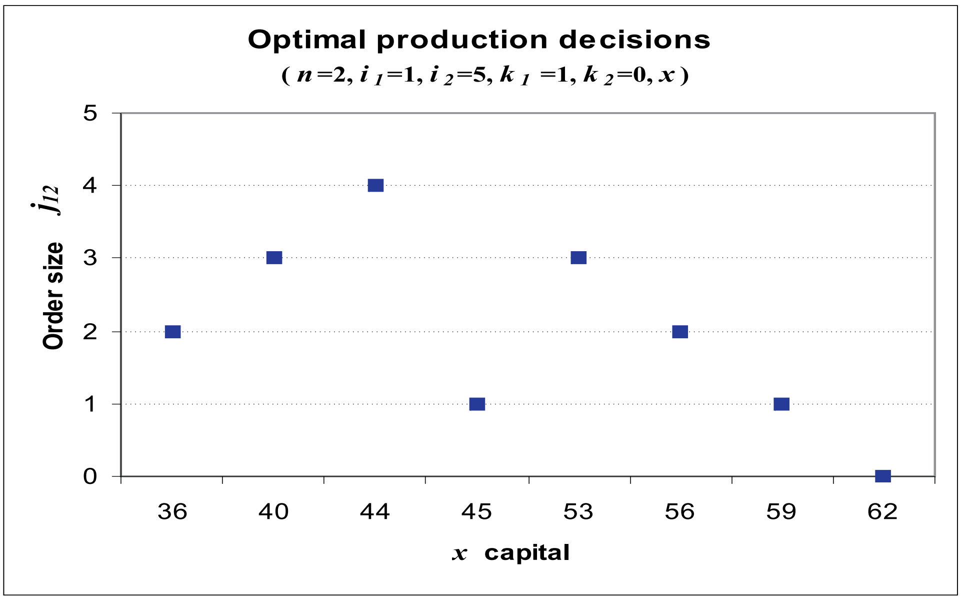
It follows from equation 1 since the bound on J_{11}, J_{22}, and J_{12} in the one lead time are less than or equal to the zero lead time.

□

Example 4.- Full substitution not necessarily optimal even if $S_2 > C_1$, and j_{12} is non monotonic with respect to i_1 and i_2.

Suppose that $p(d_1, d_2) = p'(d_1)p'(d_2)$ where $p'(d) \sim$ Poisson(5.5), $M_1 = M_2 = 20$, and $H = 32$. Let $C_1 = 1$, $S_1 = 3$, $C_2 = 2$, $S_2 = 7$. Note that the firm makes a profit selling product 1 even if component 2 is utilised to substitute for component 1. In Figure 4 we show results for state $(n = 2, i_1 = 1, i_2 = 5, x)$ where $k_1 = 1$ and $k_2 = 0$, with $d_1 = 10$, and $d_2 = 1$ where we plot the optimal production decision j_{12} with varying x.

Figure 4: j_{12} not monotonic in x



The reason we only plot the result for certain values of x, is that it is only for these values that the optimal ordering decision is $k_1 = 0$, $k_2 = 0$. For other values of x the ordering decision is different.

First note how j_{12} is not necessarily equal to J_{12}. Hence, even if $S_2 > C_1$ in some cases it is worth saving some stock for later periods, rather than using it to meet unsatisfied demand of product 1, which is not the case in zero lead time as Theorem 5 proved. Note as well how the decision is not monotonic with respect to x.

On the other hand Figure 5 shows the decision j_{12} with varying levels of i_1 in state for $n = 2$, $d_1 = 20$, $d_2 = 1$, $i_2 = 5$, and $x = 5$. This verfies that j_{12} is not monotonic with respect to i_1.

Figure 5: j_{12} not monotonic in i_1

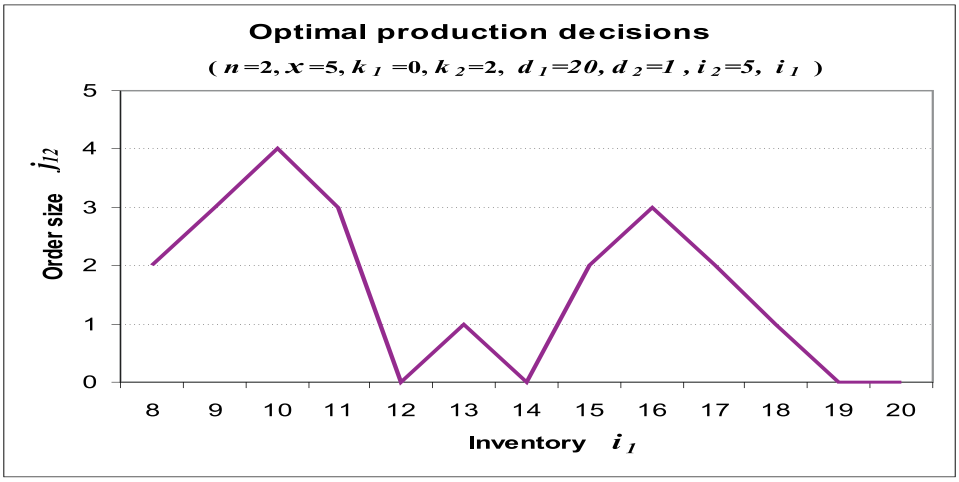
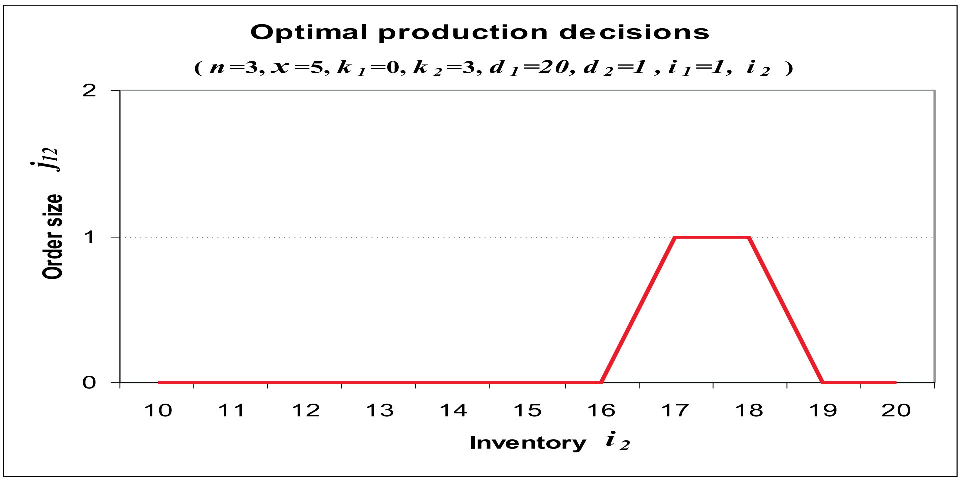


Figure 6 plots j_{12} with varying levels of i_2 in state $n = 3$, $i_1 = 1$, $k_1 = 0$, $k_2 = 3$, $x = 5$, $i_1 = 1$, with $d_1 = 20$ and $d_2 = 2$, we get, which verifies that j_{12} is not monotonic with respect to i_2.

Figure 6: j_{12} not monotonic in i_2



Hence, the properties of Corollary 1 are not valid for the zero lead time.

6 Conclusions

We have extended the results for the one product model of Archibald (2002) and Possani (2003) to consider two products, and showed that the basic properties still hold for the survival probability, stating minimum levels of capital and inventory to survive. We derived conditions under which simplified production decision can be taken when flexible components are available.

We have shown that commonality does have a positive impact on the survival probability of the firm. Moreover, we found that the usefulness of commonality is influenced by several factors, including the correlation of demand for the products, the order replenishment lead time, and the relative costs of the products and components.

The results of Section 3 suggest that commonality is very useful when the products are strongly substitutable in a fairly stable market, which lead to negative correlations in the demands. Alternatively if there is a strong positive correlation with balanced inventory levels, commonality is of little use under the survival probability objective.

The results of Section 4 and 5 showed that commonality is preferred under shorter replenishment orders. Cutting down times until replenishment makes it worthwhile to risk the component substitution (sacrifice the profit). When the substitute component is more expensive the occasions when we might want to use it become less frequent. However, there are times when we need to substitute (use commonality), even if there is a loss on the sale of the item, particularly if there is a shortage of cash to pay for the fixed overheads. Alternatively, if enough capital is available we will never want to employ an expensive component to substitute for cheaper ones. The production policy under the survival probability objective is not a simple one. It is worth noticing that simplified production decision (e.g. full substitution and monotonicity in i_1 and i_2) are harder to obtain for the 1 lead time.

Although the model we presented is a simple one it provides a framework and gives insight about the relationship between inventory, production and financial decisions in a start-up manufacturing environment.

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